Group members:

Warm-up: write the formula for the variance $\text{Var}[X]$ of a continuous random variable X below. Explain what each piece of the formula means in relation to X.

Problem 1. Compute the variance of a continuous exponential random variable X with $\lambda = 10$ on $[0, \infty)$. What proportion of the outcomes, i.e. the x-values in $[0, \infty)$, fall within 1 standard deviation of the mean?

Taylor Polynomials

Motivation: polynomials are *much easier* to work with than any other type of function – they are easy to evaluate, take limits of, differentiate, integrate and graph. The fundamental philosophy of Taylor polynomials is that *every continuous function can be approximated by polynomials to an arbitrary degree of precision.* (And ultimately, taking the limit of this polynomial approximation technique will produce a power series, which we'll learn about soon.)

Take a continuous function $f(x)$ and a point a in the domain of $f(x)$ for which we already know $f(a)$. The idea is we might want to estimate values nearby, such as $f\left(a - \frac{1}{2}\right)$, $f(a+0.01)$, $f(a-0.0004)$, without knowing their actual values. A good first estimate is the *tangent line*:

$$y = T_1(x) = f(a) + f'(a)(x-a).$$

Show that $T_1(x)$ has: (i) the same y-value as $f(x)$ at $x = a$, and (ii) the same slope as $f(x)$ at $x = a$.

This means $T_1(x)$ can be used to give an okay approximation of values $f(x)$ close to $f(a)$.

What if we want a polynomial that also matches the *concavity* of $f(x)$ at $x = a$? Try:

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

Show that $T_2(x)$ has: (i) the same y-value, (ii) the same slope and (iii) the same concavity as $f(x)$ at $x = a$.

Definition. For $n \geq 0$, the **degree** n **Taylor polynomial** approximating $f(x)$ at $x = a$ is the polynomial

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

FACT: $T_n(x)$ matches the values of $f(x)$ and the first n derivatives $f'(x), f''(x), \ldots, f^{(n)}(x)$ at $x = a$.

Problem 2. Find the degree 4 Taylor polynomial $T_4(x)$ approximating $f(x) = e^x$ at $a = 0$. Can you guess what the formula for $T_n(x)$ is?

Problem 3. Compute the first 3 digits after the decimal in the number $e = 2.???\ldots$. Then describe how you would find the first 10 digits.

Problem 4. Estimate $\sin(1)$ using the degree 5 Taylor polynomial of $f(x) = \sin(x)$ at $a = 0$. How accurate is your estimate?